Meeting 12 Math 22

Final Exam

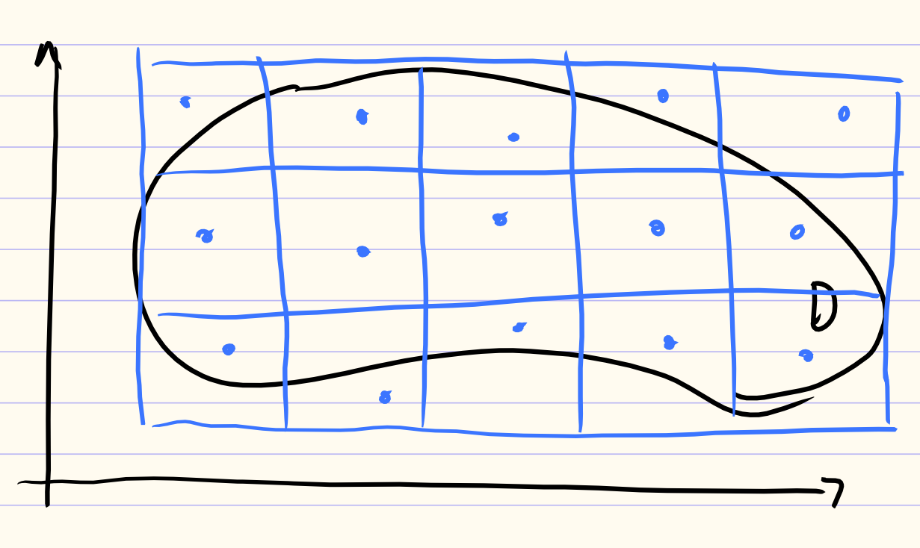
- Released tomorrow 7/22 at 3:30pm.
- Closes at 11:59 pm on Friday 7/23.
- • 3 hours to answer questions } Start by 8:00 pm
  • 45 min to submit your work } on 7/23
- Problems are from Sections 10.3 - 11.4.

Edfinity • Due on Friday at 11:59 pm.

- 33/35 responded to Sets survey so I will add 10 pts to everyones score.

Section 11.4.1 Mass of a Lamina

A lamina is a flat thin plate. We can model a lamina w/ a closed and bounded region D. Suppose that $\delta(x,y)$ gives the density of the lamina D at the point (x,y).



Cover D w/ a rectangle and divide that rectangle into $m \cdot n$ rectangles each w/ constant area ΔA. Choose test points (x_{ij}^*, y_{ij}^*) in each subrectangle and define $\delta(x_{ij}^*, y_{ij}^*) = 0$ if (x_{ij}^*, y_{ij}^*) is not on the lamina. Since density $= \frac{\text{mass}}{\text{area}}$, the mass of each small piece of the lamina is approximately

$$\delta(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

To approximate the mass of D, add all these together.

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \delta(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

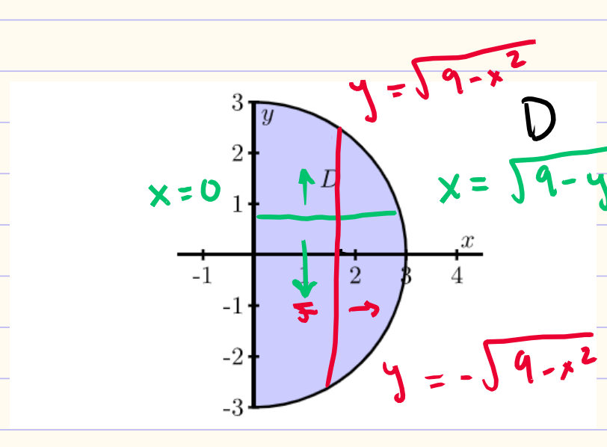
Since this is a Riemann sum, taking $m,n \to \infty$ we obtain the exact mass of the lamina

$$\iint_D \delta(x,y)\,dA.$$

Activity 11.4.2

- Complete Activity 11.4.2 and discuss w/ your group.
- Class discussion.

Find the exact mass of the following lamina w/ density $\delta(x,y) = x+y$:



We need to set up and compute $\iint_D \delta(x,y)\,dA$.

As a Type II region:

$$\iint_D \delta(x,y)\,dA = \int_{-3}^{3}\int_0^{\sqrt{9-y^2}} x+y \,dx\,dy$$
$$= \int_{-3}^{3} \left[\tfrac{1}{2}x^2 + xy\right]_0^{\sqrt{9-y^2}} dy$$
$$= \int_{-3}^{3} \tfrac{1}{2}(9-y^2) + y\sqrt{9-y^2}\,dy$$
$$= \tfrac{1}{2}\int_{-3}^{3} 9-y^2\,dy + \int_{-3}^{3} \underline{y\sqrt{9-y^2}}\,dy$$
$$= \tfrac{1}{2}\cdot 2\int_0^3 9-y^2\,dy + \underset{\uparrow}{0}$$
$$= \left[9y - \tfrac{1}{3}y^3\right]_0^3$$
$$= 9\cdot 3 - \tfrac{27}{3}$$
$$= 18.$$

odd function graph is symmetric about y = x

As a Type I region:

$$\iint_D \delta(x,y)\,dA = \int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x+y \,dy\,dx.$$

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Section 11.4.3 Area

In Calc I, you learned how to compute areas of Type I and Type II regions using single integrals.

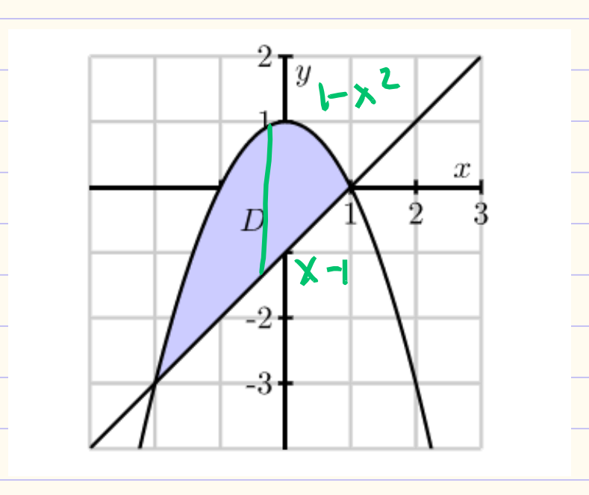
The area of D is given by

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x)-g(x)\,dx.$$

We can do it with a double integral as well.

Activity 11.4.3

- Complete Activity 11.4.3 and discuss w/ your group.
- Class discussion.

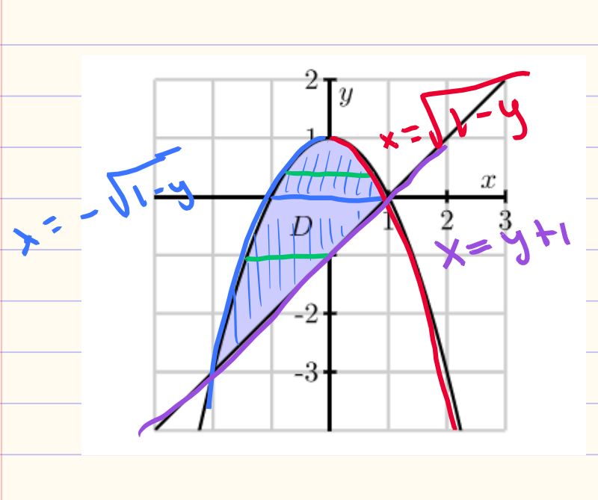


a. Set up the integral $\iint_D 1\,dA$.

As a Type I Region:

$$\iint_D 1\,dA = \int_{-2}^{1}\int_{x-1}^{1-x^2} 1\,dy\,dx$$
$$= \int_{-2}^{1} (1-x^2)-(x-1)\,dx$$
$$= \int_{-2}^{1} 2-x-x^2\,dx$$
$$= \left[2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_{-2}^{1}$$
$$= \tfrac{9}{2}.$$

$y = 1-x^2$
$\Rightarrow x = \pm\sqrt{1-y}$



As a union of Type II regions:

$$\iint_D 1\,dA = \int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} 1\,dx\,dy + \int_{-3}^{0}\int_{-\sqrt{1-y}}^{y+1} 1\,dx\,dy.$$

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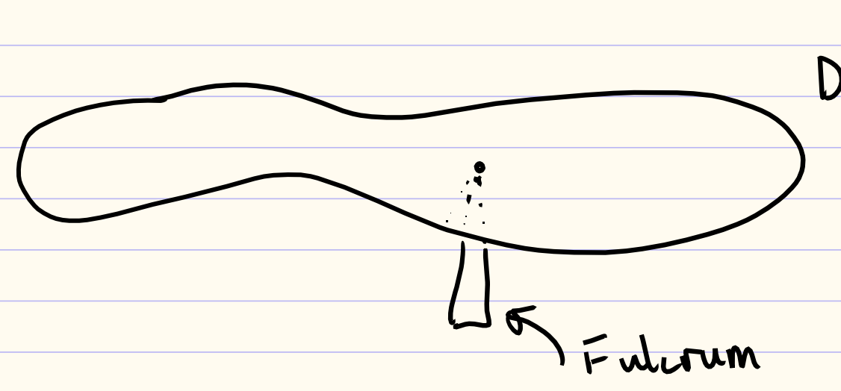
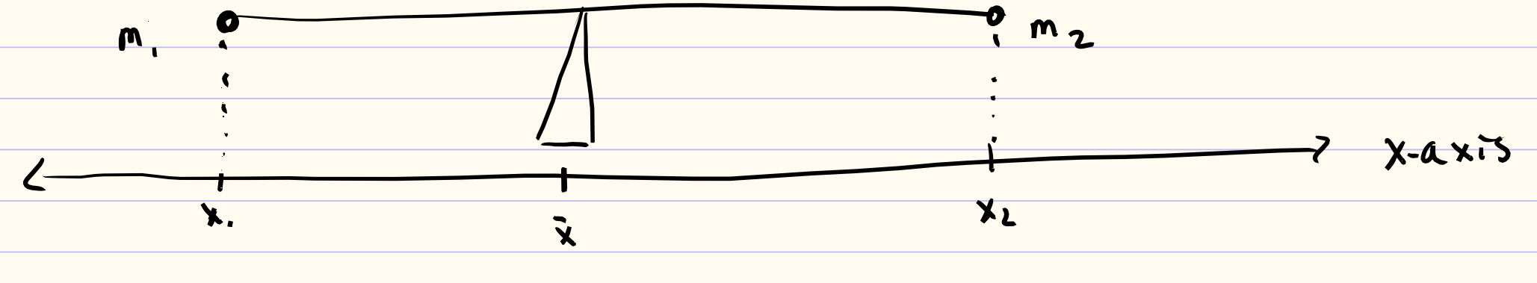
Section 11.4.3 Center of Mass

Goal: Suppose we have a lamina D with density $\delta(x,y)$. We want to find the center of mass.



This is the point $(\bar{x},\bar{y})$ in D where the lamina will balance if placed on a fulcrum.

Simplest scenario: consider two points with masses m_1, m_2 attached to the ends of a rod with negligible mass.



The rod will balance if $m_1 d_1 = m_2 d_2$. This is an expirementral fact. We want to find the x-coordinate of the fulcrum, which is the center of mass $\bar{x}$. We see that $d_1 = \bar{x} - x_1$ and $d_2 = x_2 - \bar{x}$. If $m_1 d_1 = m_2 d_2$, then

$$m_1(\bar{x}-x_1) + m_2(\bar{x}-x_2) = 0$$

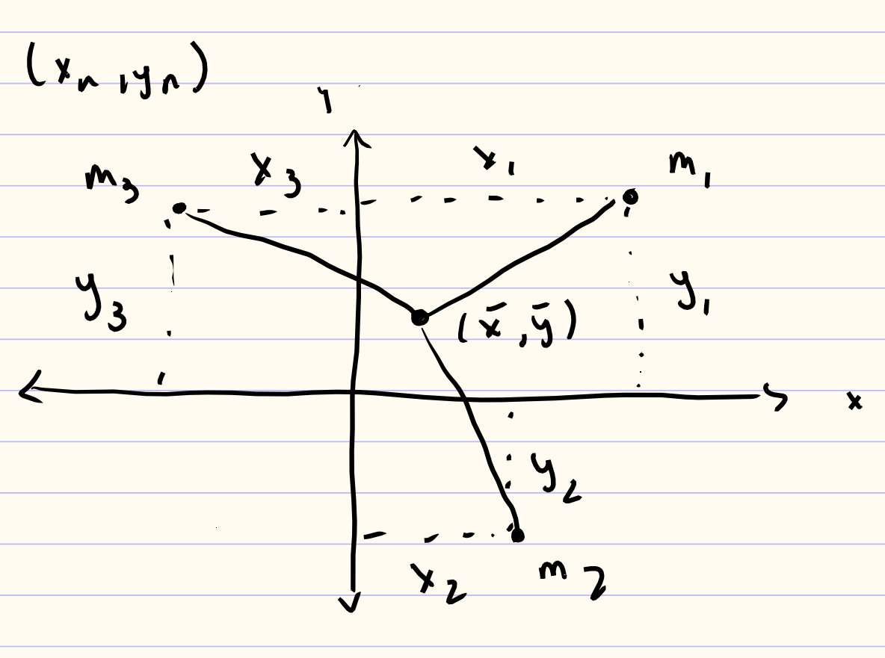
$$\Rightarrow \quad \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

In general, if we have masses $m_1, \dots, m_n$ placed along the x-axis, then the center of mass x is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}.$$

The quantities $m_i x_i$ are called the moments of the masses m_i. The sum $M = \sum_{i=1}^{n} m_i x_i$ is called the total moment. The quantity $m = \sum_{i=1}^{n} m_i$ is the total mass. The eq. becomes $m\bar{x} = M$. This says that if the total mass m were concentrated at $\bar{x}$, then the moment of that mass should be equal to the moment of the system.

Now consider n masses $m_1, \dots, m_n$ located at points $(x_1,y_1), \dots, (x_n,y_n)$



We define:

the total moment about the y axis is

$$M_y = \sum_{i=1}^{n} m_i x_i$$

the total moment about the x-axis

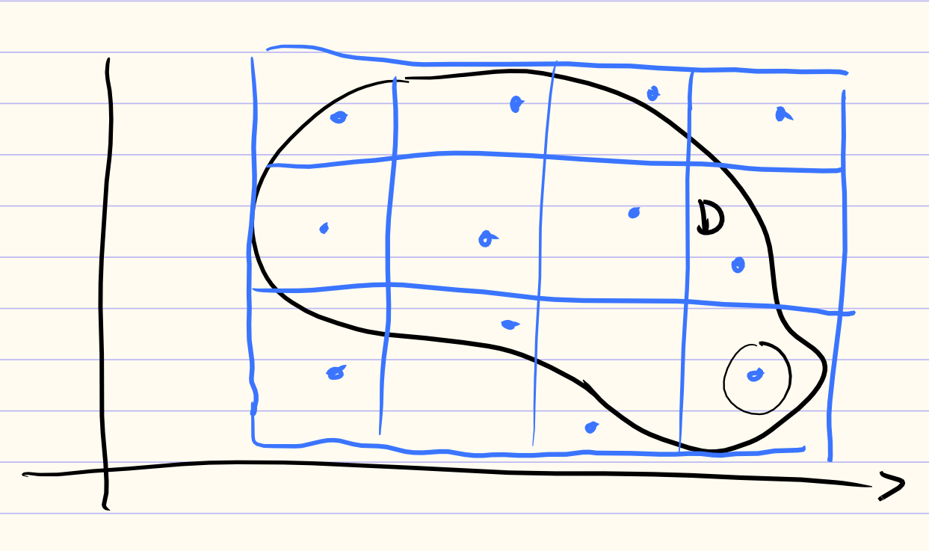
$$M_x = \sum_{i=1}^{n} m_i y_i.$$

Then the center of mass $(\bar{x},\bar{y})$ is given by

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

where $m = \sum_{i=1}^{n} m_i$.

Back to the lamina



Cover D with a rectangle and divide into $m \cdot n$ subrectangles. Choose test points (x_{ij}^*, y_{ij}^*) in each rectangle and consider each test point as a mass. The mass at (x_{ij}^*, y_{ij}^*) is approximately $\delta(x_{ij}^*, y_{ij}^*)\,\Delta A$. Then

$$\bar{x} = \frac{M_y}{m} = \frac{\sum_{j=1}^{m}\sum_{i=1}^{n} x_{ij}^*\,\delta(x_{ij}^*, y_{ij}^*)\,\Delta A}{\sum_{j=1}^{m}\sum_{i=1}^{n} \delta(x_{ij}^*, y_{ij}^*)\,\Delta A}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\sum_{j=1}^{m}\sum_{i=1}^{n} y_{ij}^*\,\delta(x_{ij}^*, y_{ij}^*)\,\Delta A}{\sum_{j=1}^{m}\sum_{i=1}^{n} \delta(x_{ij}^*, y_{ij}^*)\,\Delta A}.$$

These are all Riemann sums. Taking limits, we find that the exact center of mass $(\bar{x},\bar{y})$ is given by

$$\bar{x} = \frac{\iint_D x\,\delta(x,y)\,dA}{\iint_D \delta(x,y)\,dA}$$

$$\bar{y} = \frac{\iint_D y\,\delta(x,y)\,dA}{\iint_D \delta(x,y)\,dA}.$$

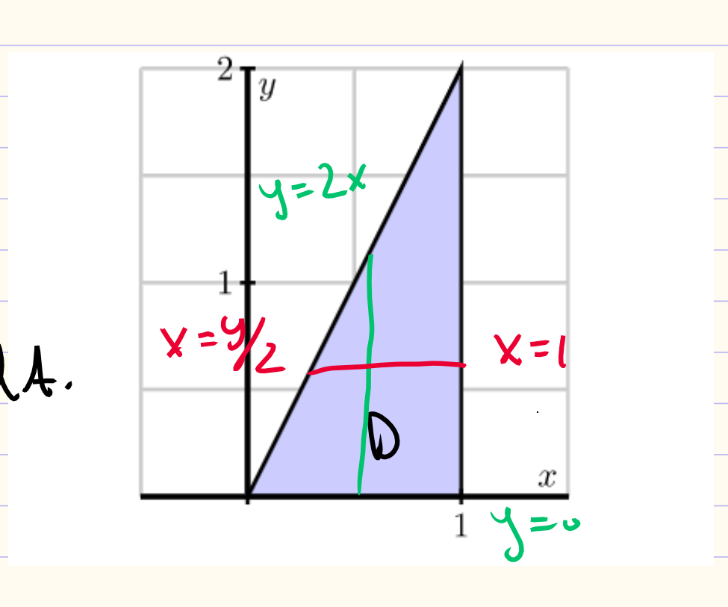
Activity 11.4.4

- Complete Activity 11.4.4 and discuss w/ your group.
- Class discussion.

The density of the lamina is given by $\delta(x,y) = 6x+6y+6$.

a. Set up an integral describing the mass m of D.

The mass of a lamina is given by $m = \iint_D \delta(x,y)\,dA$.



As a Type I Region:

$$m = \int_0^1 \int_0^{2x} 6x+6y+6\,dy\,dx$$

As a Type II Region:

$$m = \int_0^2 \int_{y/2}^{1} 6x+6y+6\,dx\,dy.$$

b. Assume $m = 14$. Recall the center of mass $(\bar{x},\bar{y})$ is given by

$$\bar{x} = \frac{M_y}{m} = \frac{1}{14}\iint_D x\,\delta(x,y)\,dA$$
$$= \frac{1}{14}\int_0^1\int_0^{2x} 6x^2+6xy+6x\,dy\,dx.$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{14}\iint_D y\,\delta(x,y)\,dA$$
$$= \frac{1}{14}\int_0^1\int_0^{2x} 6xy+6y^2+6y\,dy\,dx.$$

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